Algebraic methods for counting Euclidean embeddings of rigid graphs

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Abstract. The study of (minimally) rigid graphs is motivated by numerous applications, mostly in robotics and bioinformatics. A major open problem concerns the number of embeddings of such graphs, up to rigid motions, in Euclidean space. We capture embeddability by polynomial systems with suitable structure, so that their mixed volume, which bounds the number of common roots, to yield interesting upper bounds on the number of embeddings. We focus on $\mathbb{R}^2$ and $\mathbb{R}^3$, where Laman graphs and 1-skeleta of convex simplicial polyhedra, respectively, admit inductive Henneberg constructions. We establish the first general lower bound in $\mathbb{R}^3$ of about 2.52^n, where n denotes the number of vertices. Moreover, our implementation yields upper bounds for $n \leq 10$ in $\mathbb{R}^2$ and $\mathbb{R}^3$, which reduce the existing gaps, and tight bounds up to $n = 7$ in $\mathbb{R}^3$.

Keywords: rigid graph, Euclidean embedding, Henneberg construction, polynomial system, root bound, cyclohexane caterpillar

1 Introduction

Rigid graphs (or frameworks) constitute an old but still active area of research due to certain deep mathematical questions, as well as numerous applications, e.g. mechanism theory [8, 9], and structural bioinformatics [4].

Given graph $G = (V, E)$ and a collection of edge lengths $l_{ij} \in \mathbb{R}_{>0}$, for $(i, j) \in E$, a *Euclidean embedding* in $\mathbb{R}^d, d \geq 1$ is a mapping of the vertices V to points in $\mathbb{R}^d$, such that l_{ij} equals the Euclidean distance between the images of the i-th and j-th vertices, for all edges $(i, j) \in E$. There is no requirement on whether the edges cross or not. G is *generically* rigid in $\mathbb{R}^d$ if, for generic edge lengths, it admits a finite number of embeddings in $\mathbb{R}^d$, modulo rigid motions. G is *minimally* rigid if it is no longer rigid once any edge is removed. In the sequel, generically minimally rigid graphs are referred to as rigid.

A graph is called Laman if $|E| = 2|V| - 3$ and, additionally, all of its induced subgraphs on $k < |V|$ vertices have $\leq 2k - 3$ edges. The Laman graphs are precisely the rigid graphs in $\mathbb{R}^2$; they also admit inductive constructions. In $\mathbb{R}^3$ there is no analogous combinatorial characterization of rigid graphs, but the 1-skeleta, or edge graphs, of (convex) simplicial polyhedra are rigid in $\mathbb{R}^3$, and admit inductive constructions.
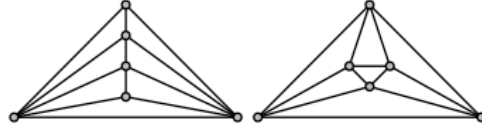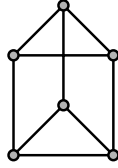
Fig. 1. The Desargues graph.   **Fig. 2.** All simplicial polyhedra for $n = 6$.

We deal with the problem of computing the maximum number of distinct planar and spatial Euclidean embeddings of rigid graphs, up to rigid motions, as a function of the number of vertices. To study upper bounds, we define a square polynomial system, expressing the edge length constraints, whose real solutions correspond precisely to the different embeddings. Here is a system expressing embeddability in $\mathbb{R}^3$, where (x_i, y_i, z_i) are the coordinates of the i-th vertex, and 3 vertices are fixed to discard translations and rotations:

$$\begin{cases} x_i = a_i, \ y_i = b_i, \ z_i = c_i, & i = 1, 2, 3, \quad a_i, b_i, c_i \in \mathbb{R}, \\ (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = l_{ij}^2, & (i, j) \in E - \{(1,2), (1,3), (2,3)\} \end{cases} \quad (1)$$

All nontrivial equations are quadratic; there are $2n - 4$ for Laman graphs, and $3n - 9$ for 1-skeleta of simplicial polyhedra, where n is the number of vertices. Bézout's bound on the number of complex roots equals the product of the degrees, hence 4^{n-2} and 8^{n-3}, respectively.

For the planar and spatial case, the best upper bounds are $\binom{2n-4}{n-2}$ and $\frac{2^{n-3}}{n-2}\binom{2n-6}{n-3}$ [2]. In applications, it is crucial to know the number of embeddings for small n. The main result in this direction was to show that the Desargues (or 3-prism) graph (Figure 1) admits 24 embeddings in $\mathbb{R}^2$ [2]. This led the same authors to lower bounds in $\mathbb{R}^2$: $24^{\lfloor (n-2)/4 \rfloor} \simeq 2.21^n$ by a "caterpillar" constructed by concatenating copies of the Desargues graph, and $2 \cdot 12^{\lfloor (n-3)/3 \rfloor} \simeq 2.29^n/6$ obtained by a Desargues "fan" [4].

Bernstein's bound on the number of roots of a polynomial system exploits the sparseness of the equations. It is bounded by Bézout's bound and typically much tighter. We have developed software to construct all rigid graphs up to isomorphism, for small n, and compute the Bernstein's bounds. Besides some straightforward upper bounds in Lemmas 1 and 6, our main contribution is twofold. We derive the first general lower bound in $\mathbb{R}^3$:

$$16^{\lfloor (n-3)/3 \rfloor} \simeq 2.52^n, \quad n \geq 9,$$

by designing a cyclohexane caterpillar. We also obtain improved upper and lower bounds for $n \leq 10$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ (Tables 1 and 2). Moreover, we establish tight bounds for $n \leq 7$ in $\mathbb{R}^3$ by appropriately formulating the polynomial system. We apply Bernstein's Second theorem to show that the above polynomial system cannot yield tight bounds.

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[4] We have corrected the exponent of the original statement.

The rest of the paper is structured as follows: Section 2 discusses the planar case, Section 3 presents our algebraic tools, Section 4 deals with $\mathbb{R}^3$, and we conclude with open questions.

Some results appeared in [5] in preliminary form.

2 Planar embeddings of Laman graphs

Laman graphs admit inductive constructions, starting with a triangle, and followed by a sequence of Henneberg-1 (or H_1) and Henneberg-2 steps (or H_2). Each step adds a new vertex: H_1 connects it to two existing vertices, H_2 connects it to 3 existing vertices having at least one edge among them, which is removed. We represent a Laman graph by its Henneberg sequence $\triangle s_4 \ldots, s_n$, where $s_i \in \{1, 2\}$. A Laman graph is called H_1 if and only if it can be constructed using only H_1; otherwise it is called H_2. Since two generic circles intersect in two real points, H_1 exactly doubles the maximum number of embeddings. It follows that a H_1 graph has 2^{n-2} embeddings.

One can easily verify that every $\triangle 2$ graph is isomorphic to a $\triangle 1$ graph and that every $\triangle 12$ graph is isomorphic to a $\triangle 11$ graph. Consequently, all Laman graphs with $n = 4, 5$ are H_1, with 4 and 8 embeddings, respectively. For $n = 6$, there are 3 possibilities: the graph is either H_1, $K_{3,3}$, or the Desargues graph. Since the $K_{3,3}$ graph has at most 16 embeddings [8, 9] and the Desargues graph has 24 embeddings [2], the upper bound is 24.

Using our software (Section 3), we construct all Laman graphs for $n = 7, \ldots, 10$, compute the respective Bernstein's bounds, and bound the maximum number of Euclidean embeddings by is 64, 128, 512 and 2048, respectively. Table 1 summarizes our results for $n \leq 10$. The lower bound for $n = 9$ follows from the Desargues fan. All other lower bounds follow from the fact that H_1 doubles the number of embeddings.

We now establish a "sparse" upper bound.

Lemma 1. *Let G be Laman, with $k \geq 4$ degree-2 vertices. Then, the number of its planar embeddings is at most $2^{k-4}4^{n-k}$.*

Proof. There are at least k removals of degree-2 vertices, since all created graphs are Laman. The final graph has a Bézout bound of 4^{n-k}.

3 An algebraic interlude

Given a polynomial f in n variables, its support is the set of exponents in $\mathbb{N}^n$ corresponding to nonzero terms (or monomials). The Newton polytope of f is the convex hull of its support and lies in $\mathbb{R}^n$. Consider polytopes $P_i \subset \mathbb{R}^n$ and $\lambda_i \in \mathbb{R}, \lambda_i \geq 0$, for $i = 1, \ldots, n$. Consider the Minkowski sum $\lambda_1 P_1 + \cdots + \lambda_n P_n \in \mathbb{R}^n$: its (Euclidean) volume is a homogeneous polynomial of degree n in the λ_i. The coefficient of $\lambda_1 \cdots \lambda_n$ is the *mixed volume* of $P_1, \ldots, P_n$. If $P_1 = \cdots = P_n$, then the mixed volume is $n!$ times the volume of P_1. We focus on $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Theorem 2. [1] *Let $f_1 = \cdots = f_n = 0$ be a polynomial system in n variables with real coefficients, where the f_i have fixed supports. The number of isolated common solutions in $(\mathbb{C}^*)^n$ is bounded above by the mixed volume of (the Newton polytopes of) the f_i. This bound is tight for generic coefficients of the f_i's.*

Bernstein's Second theorem below was used in $\mathbb{R}^2$ [7]; we apply it to $\mathbb{R}^3$. Given $v \in \mathbb{R}^n - \{0\}$ and polynomial f_i, $\partial_v f_i$ is the polynomial obtained by keeping only the terms whose exponents minimize inner product with v; its Newton polytope is the face of the Newton polytope of f_i supported by v.

Theorem 3. [1] *If for all $v \in \mathbb{R}^n - \{0\}$ the face system $\partial_v f_1 = \ldots = \partial_v f_n = 0$ has no solutions in $(\mathbb{C}^*)^n$, then the mixed volume of the f_i exactly equals the number of solutions in $(\mathbb{C}^*)^n$, and all solutions are isolated. Otherwise, the mixed volume is a strict upper bound.*

In order to bound the number of embeddings of rigid graphs, we have developed specialized software that constructs all Laman graphs and all 1-skeleta of simplicial polyhedra with $n \leq 10$. Our computational platform is SAGE [5]. We construct all graphs using the Henneberg steps, which we implemented in Python, using SAGE's interpreter. We classify all graphs up to isomorphism using SAGE's interface with N.I.C.E., an open-source isomorphism check software, keeping, for each graph, the Henneberg sequence with the fewest H_1 steps.

For each graph we construct a system whose real solutions express all embeddings, by formulation (2). We bound the number of its (complex) solutions by mixed volume. For every Laman graph, to discard translations and rotations, we fix one edge to be of unit length, aligned with an axis, with one vertex at the origin. In $\mathbb{R}^3$, a third vertex is fixed in a coordinate plane. Depending on the choice of the fixed edge, we obtain different systems hence different mixed volumes, and we keep their minimum.

We used an Intel Core2, at 2.4GHz, with 2GB of RAM. We tested more that $20,000$ graphs, computed the mixed volume of more than $40,000$ systems, taking a total time of about 2 days. Tables 1 and 2 summarize our results.

4 Spatial embeddings of 1-skeleta of simplicial polyhedra

Let us examine 1-skeleta of (convex) simplicial polyhedra, which are rigid in $\mathbb{R}^3$ [6]. For such a graph (V, E), $|E| = 3|V| - 6$ and all induced subgraphs on $k < |V|$ vertices have $\leq 3k - 6$ edges. Consider any $k + 2$ vertices forming a cycle with $\geq k - 1$ diagonals, $k \geq 1$. The extended Henneberg-k step (or H_k), $k = 1, 2, 3$, corresponds to adding a vertex, connecting it to the $k + 2$ vertices, and removing $k - 1$ diagonals among them. A graph is the 1-skeleton of a simplicial polyhedron in $\mathbb{R}^3$ if and only if it has a construction starting with the 3-simplex, followed by any sequence of H_1, H_2, H_3 [3].

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[5] http://www.sagemath.org/

Since 3 spheres intersect generically in two points, H_1 exactly doubles the maximum number of embeddings. In order to discard translations and rotations, we fix a (triangular) facet of the polytope; we choose, without loss of generality, the first 3 vertices and obtain system (1) of dimension $3n$. Let $v = (0,0,0,0,0,0,0,0,0,-1,\ldots,-1) \in \mathbb{R}^{3n}$, the face system is:

$$\begin{cases} x_i = a_i, y_i = b_i, z_i = c_i, & i = 1,2,3, \quad a_i, b_i, c_i \in \mathbb{R}, \\ (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = 0, & (i,j) \in E, \ i,j \notin \{1,2,3\}, \\ x_i^2 + y_i^2 + z_i^2 = 0, & (i,j) \in E : \ i \notin \{1,2,3\}, \ j \in \{1,2,3\}. \end{cases}$$

This system has $(a_1, b_1, c_1, \ldots, a_3, b_3, c_3, 1, 1, \gamma\sqrt{2}, \ldots, 1, 1, \gamma\sqrt{2}) \in (\mathbb{C}^*)^{3n}$ as a solution, where $\gamma = \pm\sqrt{-1}$. According to Theorem 3, the mixed volume is not a tight bound on the number of solutions in $(\mathbb{C}^*)^{3n}$. This was also observed, for $\mathbb{R}^2$, in [7]. To remove spurious solutions let $w_i = x_i^2 + y_i^2 + z_i^2$, for $i = 1, \ldots, n$. This yields an equivalent system, with lower mixed volume, which will be used in our computations:

$$\begin{cases} x_i = a_i, y_i = b_i, z_i = c_i, & i = 1,2,3, \\ w_i = x_i^2 + y_i^2 + z_i^2, & i = 1, \ldots, n, \\ w_i + w_j - 2x_i x_j - 2y_i y_j - 2z_i z_j = l_{ij}^2, & (i,j) \in E - \{(1,2),(1,3),(2,3)\}. \end{cases} \tag{2}$$

For $n = 4$, the only simplicial polytope is the 3-simplex, which has 2 embeddings. For $n = 5$, there is a unique 1-skeleton of a simplicial polyhedron [3], and is obtained from the 3-simplex by H_1, hence it has exactly 4 embeddings.

For $n = 6$, there are two non-isomorphic graphs G_1, G_2 (Figure 2) [3], yielding respective mixed volumes of 8 and 16. G_2 is the graph of the cyclohexane, which admits 16 different embeddings [4]. To see this, the cyclohexane is a 6-cycle with known lengths between vertices at distance 1 (adjacent) and 2. Alternatively, G_2 corresponds to a Stewart platform parallel robot with 16 configurations, where triangles define the platform and base, and 6 lengths link the triangles in a jigsaw shape. This proves:

Lemma 4. *The 1-skeleton of a simplicial polyhedron with $n = 6$ has at most 16 embeddings and this is tight.*

Let us glue copies of cyclohexanes sharing a triangle, each adding 3 vertices, thus obtaining the 1-skeleton of a simplicial polytope. Applying Lemma 4 we have:

Theorem 5. *There exist edge lengths for which the cyclohexane caterpillar construction has $16^{\lfloor (n-3)/3 \rfloor} \simeq 2.52^n$ embeddings, for $n \geq 9$.*

Table 2 summarizes our results for $n \leq 10$. The upper bounds for $n = 7, \ldots, 10$ are computed by our software. The lower bound for $n = 9$ is from Theorem 5. All other lower bounds are obtained by considering a H_1 construction.

We state without proof a result similar to Lemma 1.

Lemma 6. *Let G be the 1-skeleton of a simplicial polyhedron with $k \geq 9$ degree-3 vertices. The number of embeddings of G is bounded above by $2^{k-9} 8^{n-k}$.*

n =	3	4	5	6	7	8	9	10
lower	2	4	8	24	48	96	288	576
upper	2	4	8	24	64	128	512	2048

Table 1. Bounds for Laman graphs.

n =	4	5	6	7	8	9	10
lower	2	4	16	32	64	256	512
upper	2	4	16	32	160	640	2560

Table 2. Bounds for 1-skeleta of simplicial polyhedra.

5 Further work

The most important problem in rigidity theory is the combinatorial characterization of rigid graphs in $\mathbb{R}^3$. Since we rely on Henneberg constructions, it is crucial to determine the effect of each step on the number of embeddings: we conjecture that H_2 multiplies it by ≤ 4 and H_3 by ≤ 8, but these may not always be tight. Our conjecture has been verified for small n.

References

1. D.N. Bernstein. The number of roots of a system of equations. *Fun. Anal. Pril.*, 9:1–4, 1975.
2. C. Borcea and I. Streinu. The number of embeddings of minimally rigid graphs. *Discrete Comp. Geometry*, 31(2):287–303, 2004.
3. R. Bowen and S. Fisk. Generation of triangulations of the sphere. *Math. of Computation*, 21(98):250–252, 1967.
4. I.Z. Emiris and B. Mourrain. Computer algebra methods for studying and computing molecular conformations. *Algorithmica, Special Issue*, 25:372–402, 1999.
5. I.Z. Emiris and A. Varvitsiotis. Counting the number of embeddings of minimally rigid graphs. In *Proc. Europ. Workshop Comput. Geometry*, Brussels, 2009.
6. H. Gluck. Almost all simply connected closed surfaces are rigid. *Lect. Notes in Math.*, 438:225–240, 1975.
7. R. Steffens and T. Theobald. Mixed volume techniques for embeddings of Laman graphs. In *Proc. Europ. Workshop Comput. Geometry*, pages 25–28, Nancy, France, 2008. Final version accepted in *Comp. Geom: Theory & Appl., Special Issue*.
8. D. Walter and M. Husty. On a 9-bar linkage, its possible configurations and conditions for paradoxical mobility. In *IFToMM Congress*, Besançon, France, 2007.
9. W. Wunderlich. Gefärlice Annahmen der Trilateration und bewegliche Fachwerke I. *Zeitschrift für Angewandte Mathematik und Mechanik*, 57:297–304, 1977.